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DAAG29-81-K-0032

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Unclassified  
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 17902.14-MA	2. GOVT ACCESSION NO. N/A AD-A114 252	3. RECIPIENT'S CATALOG NUMBER N/A
4. TITLE (and Subtitle) A Note on the Multiple Scale Fourier Transform		5. TYPE OF REPORT & PERIOD COVERED Reprint
7. AUTHOR(s) A. Jeffrey T. Kawahara		6. PERFORMING ORG. REPORT NUMBER N/A
9. PERFORMING ORGANIZATION NAME AND ADDRESS Stanford University Stanford, CA 94305		8. CONTRACT OR GRANT NUMBER(s) DAAG29 81 K 0032
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P. O. Box 12211 Research Triangle Park, NC 27709		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS N/A
12. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		12. REPORT DATE 1981
		13. NUMBER OF PAGES 10
		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report)  Submitted for announcement only.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)  A		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number)		

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## A NOTE ON THE MULTIPLE SCALE FOURIER TRANSFORM

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(Received 1 April 1981)

*Key words:* Perturbation, multiple scales, Fourier transform, nonlinear ODEs and PDEs.

### 1. INTRODUCTION

IN A RECENT PAPER, the multiple scale Fourier transformation has been applied to describe the far-field asymptotic behaviour of nonlinear dispersive waves (Jeffrey and Kawahara [1]). It was shown that the use of the multiple scale Fourier transformation can systematize a nonlinear asymptotic perturbation analysis, and that the perturbation analysis can then be reduced to simple manipulations with respect to Dirac delta functions involving the multiple scale wave-number and frequency space.

In the present paper, we apply the multiple scale Fourier transformation to oscillation and wave problems which include *damping* or *dissipative* effects. When these effects are included in a system, the solution in the physical space may involve exponential changes, but exponential functions do not allow a Fourier transformation in the ordinary sense. However, we can introduce a Fourier transformation for exponential functions if we extend the notion of a Fourier transformation to include generalized functions. A Fourier transformation of an exponential function can then be represented by means of a delta function in a complex Fourier space (Bremermann [2], Challifour [3]). Therefore, in terms of an appropriate definition of the delta function in a complex Fourier space, we can proceed with a perturbation analysis by means of simple manipulations with respect to delta functions, even though the system under consideration may include damping or dissipative effects.

As a simple illustrative example, we consider the perturbation approximation of a linear oscillation problem involving small damping. The essential idea can be fully explained in terms of this example, and thereafter applications to general problems will be a straightforward matter.

In Section 2, we start by solving the problem in terms of the derivative expansion method in order to make clear the correspondence between the physical space representation and the

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Fourier space representation. In Section 3, we show how the Fourier transformation should be extended so as to include exponential changes. Once the extension is introduced, the calculations become straightforward, as is shown in the latter half of Section 3. Some final remarks are made in Section 4, and an introduction to the notion and use of a delta function in a complex space is given in the Appendices.

## 2. SOLUTION BY MEANS OF THE DERIVATIVE EXPANSION METHOD

We consider the following linear oscillation problem involving small damping

$$\frac{d^2 f}{dt^2} + 2\varepsilon \frac{df}{dt} + f = 0, \quad (2.1)$$

where the initial conditions are given by

$$f(0) = a, \quad \frac{df(0)}{dt} = 0. \quad (2.2)$$

This problem admits the exact solution

$$f(t; \varepsilon) = a e^{-\varepsilon t} [\cos(1 - \varepsilon^2)^{1/2} t + \varepsilon(1 - \varepsilon^2)^{-1/2} \sin(1 - \varepsilon^2)^{1/2} t]. \quad (2.3)$$

First of all, let us solve this problem in terms of the derivative expansion method. Introducing the multiple time scales  $t_m = \varepsilon^m t$  ( $m = 0, 1, \dots, M$ ) and the expansions

$$f(t; \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n f_n(t_0, t_1, \dots, t_M), \quad (2.4)$$

$$\frac{d}{dt} \equiv \sum_{m=0}^M \varepsilon^m \frac{\partial}{\partial t_m}, \quad (2.5)$$

into (2.1) and (2.2), we find the set of perturbation equations

$$\frac{\partial^2 f_0}{\partial t_0^2} + f_0 = 0, \quad (2.6a)$$

$$\frac{\partial^2 f_1}{\partial t_0^2} + f_1 + 2 \frac{\partial^2 f_0}{\partial t_0 \partial t_1} + 2 \frac{\partial f_0}{\partial t_0} = 0, \quad (2.6b)$$

$$\begin{aligned} \frac{\partial^2 f_2}{\partial t_0^2} + f_2 + 2 \frac{\partial^2 f_1}{\partial t_0 \partial t_1} + 2 \frac{\partial^2 f_1}{\partial t_0 \partial t_2} + \frac{\partial^2 f_0}{\partial t_1^2} \\ + 2 \frac{\partial f_1}{\partial t_0} + 2 \frac{\partial f_0}{\partial t_1} = 0, \quad \dots \end{aligned} \quad (2.6c)$$

and the boundary conditions

$$f_0(0, \dots, 0) = a, \quad f_1(0, \dots, 0) = f_2(0, \dots, 0) = 0, \quad (2.7a)$$

$$\frac{\partial f_0(0, \dots, 0)}{\partial t_0} = 0, \quad \frac{\partial f_1(0, \dots, 0)}{\partial t_0} + \frac{\partial f_0(0, \dots, 0)}{\partial t_1} = 0,$$

$$\frac{\partial f_2(0, \dots, 0)}{\partial t_0} + \frac{\partial f_1(0, \dots, 0)}{\partial t_1} + \frac{\partial f_0(0, \dots, 0)}{\partial t_2} = 0, \quad \dots \quad (2.7b)$$

The solution of (2.6a) is given by

$$f_0 = A_0 e^{i\omega_0 t} + \text{c.c.}, \quad (2.8)$$

where  $A_0$  is a complex function of the slow variables  $t_1, t_2, \dots$ , and c.c. denotes the complex conjugate of the preceding term.

It follows from (2.6b) that

$$\frac{\partial^2 f_1}{\partial t_0^2} + f_1 = -2i \left( \frac{\partial A_0}{\partial t_1} + A_0 \right) e^{i\omega_0 t} + \text{c.c.} \quad (2.9)$$

The non-secularity condition for (2.9) gives

$$\frac{\partial A_0}{\partial t_1} + A_0 = 0, \quad (2.10)$$

and the non-secular solution is then given by

$$f_1 = A_1 e^{i\omega_0 t} + \text{c.c.} \quad (2.11)$$

Proceeding to (2.6c) we obtain the non-secularity condition

$$2i \left( \frac{\partial A_1}{\partial t_1} + A_1 \right) + \left( 2i \frac{\partial A_0}{\partial t_2} + \frac{\partial^2 A_0}{\partial t_1^2} + 2 \frac{\partial A_0}{\partial t_1} \right) = 0. \quad (2.12)$$

Solving (2.10) we get

$$A_0(t_1, t_2, \dots) = B_0(t_2, \dots) e^{-t_1}, \quad (2.13)$$

and substituting this into (2.12), we obtain

$$\frac{\partial A_1}{\partial t_1} + A_1 = - \left( \frac{\partial B_0}{\partial t_2} + \frac{i}{2} B_0 \right) e^{-t_1}. \quad (2.14)$$

The solution  $A_1$  for (2.14) becomes bounded (or  $f_1/f_0 \rightarrow 0$  as  $t \rightarrow \infty$ ) if

$$\frac{\partial B_0}{\partial t_2} + \frac{i}{2} B_0 = 0, \quad (2.15)$$

and is then given by

$$A_1(t_1, t_2, \dots) = B_1(t_2, \dots) e^{-t_1}. \quad (2.16)$$

From (2.15) we obtain

$$B_0(t_2, t_3, \dots) = C_0(t_3, \dots) e^{-it_2/2}, \quad (2.17a)$$

so that

$$A_0(t_1, t_2, \dots) = C_0(t_3, \dots) e^{-t_1 - it_2/2}. \quad (2.17b)$$

The next higher order condition for the non-secularity is found to be

$$2i \left( \frac{\partial A_2}{\partial t_1} + A_2 \right) + \left( 2i \frac{\partial A_1}{\partial t_2} + \frac{\partial^2 A_1}{\partial t_1^2} + 2 \frac{\partial A_1}{\partial t_1} \right) + 2 \left( i \frac{\partial A_0}{\partial t_3} + \frac{\partial^2 A_0}{\partial t_1 \partial t_2} + \frac{\partial A_0}{\partial t_2} \right) = 0. \quad (2.18)$$

Similar calculations are also possible for higher order terms, and up to the  $O(\varepsilon^4)$ -solution we arrive at

$$\begin{aligned} f = & E_0 e^{it_0 - t_1 - it_2/2 - it_4/8} + \text{c.c.} \\ & + \varepsilon(D_1 e^{it_0 - t_1 - it_2/2} + \text{c.c.}) \\ & + \varepsilon^2(C_2 e^{it_0 - t_1 - it_2/2} + \text{c.c.}) \\ & + \varepsilon^2(B_3 e^{it_0 - t_1} + \text{c.c.}) + \varepsilon^4(A_4 e^{it_0} + \text{c.c.}). \end{aligned} \quad (2.19)$$

The boundary conditions (2.7) determine the coefficients of (2.19) successively, giving rise to

$$\begin{aligned} E_0 &= a/2, \quad D_1 = -ia/2, \quad C_2 = 0, \\ B_3 &= -ia/4, \quad A_4 = 0, \end{aligned} \quad (2.20)$$

where  $E_0, D_1, C_2, B_3$ , and  $A_4$  are assumed to be constants with respect to the slow variables equal to or slower than  $t_5, t_4, t_3, t_2$ , and  $t_1$ , respectively. Consequently, up to the  $O(\varepsilon^4)$ -approximation calculated here, we obtain

$$\begin{aligned} f &= \frac{a}{2} e^{-t_1} \left[ e^{it_0 - it_2/2 - it_4/8} - i\varepsilon e^{it_0 - it_2/2} - \frac{i}{2} \varepsilon^3 e^{it_0} + \text{c.c.} \right] + O(\varepsilon^5) \\ &= a e^{-t_1} \left[ \cos \left( 1 - \frac{\varepsilon^2}{2} - \frac{\varepsilon^4}{8} \right) t + \varepsilon \sin \left( 1 + \frac{\varepsilon^2}{2} \right) t + \frac{\varepsilon^3}{2} \sin t \right] + O(\varepsilon^5). \end{aligned} \quad (2.21)$$

It should be noticed here that the solution (2.21) is not a simple expansion in  $\varepsilon$  of the exact solution (2.3). The expansion of the factor  $(1 - \varepsilon^2)^{1/2}$  in the exact solution reproduces exactly the terms in (2.21) up to the order of approximation calculated here. The derivative expansion that avoids the secularity thus incorporates partial sums, in the sense that the perturbation solution so obtained is not a simple power series solution in  $\varepsilon$ .

### 3. MULTIPLE SCALE FOURIER TRANSFORMATION

We shall now solve the problem (2.1) subject to (2.2) in the Fourier space. The Fourier transformation for the function  $f$  with multiple scales  $t_0, t_1, \dots, t_M$  is first introduced as follows:

$$f(t_0, t_1, \dots, t_M) = \int \int_{-\infty}^{\infty} g(\omega_0, \omega_1, \dots, \omega_M) \exp \left\{ i \sum_{m=0}^M \omega_m t_m \right\} \prod_{m=0}^M d\omega_m, \quad (3.1)$$

$$g(\omega_0, \omega_1, \dots, \omega_M) = \frac{1}{(2\pi)^{M+1}} \int \int_{-\infty}^{\infty} f(t_0, t_1, \dots, t_M) \exp \left\{ -i \sum_{m=0}^M \omega_m t_m \right\} \prod_{m=0}^M dt_m. \quad (3.2)$$

At the moment we assume that  $\omega_0, \omega_1, \dots$ , are real.

In agreement with the expansions (2.4) and (2.5), we expand the Fourier amplitude as follows:

$$g(\omega_0, \omega_1, \dots, \omega_M) = \sum_{n=0}^{\infty} \varepsilon^n g_n(\omega_0, \omega_1, \dots, \omega_M). \quad (3.3)$$

It then follows from (3.1) that the time derivative should be replaced in the multiple scale Fourier space by the operation

$$i(\omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots). \quad (3.4)$$

The introduction of (3.3) and (3.4) into (2.1) and (2.2) now yields

$$(-\omega_0^2 + 1)g_0 = 0, \quad (3.5a)$$

$$(-\omega_0^2 + 1)g_1 - 2\omega_0(\omega_1 - i)g_0 = 0,$$

$$(-\omega_0^2 + 1)g_2 - 2\omega_0(\omega_1 - i)g_1 - (\omega_1^2 + 2\omega_0\omega_2 - 2i\omega_1)g_0 = 0, \quad (3.5c)$$

while the boundary conditions reduce to

$$\int_{-\infty}^{\infty} g_0 \prod_{m=0}^M d\omega_m = a, \quad \int_{-\infty}^{\infty} g_1 \prod_{m=0}^M d\omega_m = \dots = 0, \quad (3.6a)$$

$$\begin{aligned} \int_{-\infty}^{\infty} \omega_0 g_0 \prod_{m=0}^M d\omega_m &= \int_{-\infty}^{\infty} (\omega_0 g_1 + \omega_1 g_0) \prod_{m=0}^M d\omega_m \\ &= \int_{-\infty}^{\infty} (\omega_0 g_2 + \omega_1 g_1 + \omega_2 g_0) \prod_{m=0}^M d\omega_m = \dots = 0. \end{aligned} \quad (3.6b)$$

It will be shown in the course of the calculations that follow how the necessity arises for the extension of  $\omega_0, \omega_1, \dots$ , to the complex plane.

The general solution to (3.5a) may be expressed in the form

$$g_0(\omega_0, \omega_1, \dots) = \delta(\omega_0 - 1) a_0(\omega_1, \omega_2, \dots) + \delta(\omega_0 + 1) a^*(-\omega_1, -\omega_2, \dots), \quad (3.7)$$

where we have made use of the reality of  $f_0$ , and \* denotes a complex conjugate. It is readily seen that the inverse Fourier transformation of (3.7) reproduces (2.8), if we put

$$A_0(t_1, t_2, \dots) = \int_{-\infty}^{\infty} a_0(\omega_1, \omega_2, \dots) \exp \left\{ i \sum_{m=1}^M \omega_m t_m \right\} \prod_{m=1}^M d\omega_m. \quad (3.8)$$

The substitution of (3.7) into (3.5b) leads directly to the result

$$\begin{aligned} g_1(\omega_0, \omega_1, \dots) &= -\frac{2\omega_0}{\omega_0^2 - 1} \delta(\omega_0 - 1) (\omega_1 - i) a_0(\omega_1, \omega_2, \dots) \\ &\quad - \frac{2\omega_0}{\omega_0^2 - 1} \delta(\omega_0 + 1) (\omega_1 - i) a_0^*(-\omega_1, -\omega_2, \dots). \end{aligned} \quad (3.9)$$

Now it may be seen from the expression (3.9) that  $g_1$  becomes singular at  $\omega_0 = \pm 1$ , because the denominator  $\omega_0^2 - 1$  becomes zero at the origin of the respective delta functions  $\delta(\omega_0 - 1)$  and  $\delta(\omega_0 + 1)$ .

These singularities can be eliminated if we can set

$$\begin{aligned} \delta(\omega_0 - 1) (\omega_1 - i) a_0(\omega_1, \omega_2, \dots) &= 0, \\ \delta(\omega_0 + 1) (\omega_1 - i) a_0^*(-\omega_1, -\omega_2, \dots) &= 0, \end{aligned} \quad (3.10)$$

at  $\omega_0^2 = \pm 1$ , or if we can set

$$\begin{aligned} a_0(\omega_1, \omega_2, \dots) &= \delta(\omega_1 - i) b_0(\omega_2, \dots), \\ a_0^*(-\omega_1, -\omega_2, \dots) &= \delta(\omega_1 - i) b_0^*(-\omega_2, \dots). \end{aligned} \quad (3.11)$$

So far  $\omega_0, \omega_1, \dots$ , have been assumed to be real, so that  $\omega_1$  is real, and consequently the expression  $\delta(\omega_1 - i)$  does not possess a well-defined meaning. As is easily seen from (2.10) and

(2.13), the expression  $\delta(\omega_1 - i)$  corresponds to the exponential change  $e^{-t_1}$  in the physical space. The exponential function is not absolutely integrable, and so it has no Fourier transformation in the ordinary sense. However, we can apply the Fourier transformation for generalized functions to the present example. It is then necessary to extend the Fourier space  $\omega_0, \omega_1, \dots$ , into a complex space.

It is possible to introduce the Fourier transformation of an exponential function by appeal to the theory of Fourier transformations for generalized functions (Challifour [3]). The following formulae that result are useful for the present analysis:

$$\mathcal{F}[e^{bt}] = 2\pi\delta(\omega + bi), \quad (3.12a)$$

$$\mathcal{F}[e^{iat}] = 2\pi\delta(\omega + a), \quad (3.12b)$$

$$\mathcal{F}[e^{iat} e^{bt}] = 2\pi\delta(\omega + a + bi), \quad (3.12c)$$

where  $\omega$  has been extended to a complex space, and the details of the extension are shown in Appendix A. Formula (3.12a) may be used to define  $\delta(\omega_1 - i)$  in the present problem. The relation (3.12b) indicates that a sinusoidal change in the physical space corresponds to a delta function in the Fourier space, shifted by an amount  $a$ , as already noted previously (Jeffrey and Kawahara [1]). Formula (3.12c) will be useful for other general problems, as is shown in Appendix B.

For practical calculations, it is convenient to assume merely that  $\omega_0, \omega_1, \dots$ , have all been extended to complex quantities, and thereafter to carry out calculations in terms of delta functions in a complex space. For the present problem, it suffices to introduce the complex quantity  $\omega_1$  for the delta function  $\delta(\omega_1 - i)$ , but we extend (3.1) and (3.2) to make them valid for all complex  $\omega_m$ .

The lowest order solution is then given by

$$g_0(\omega_0, \omega_1, \dots) = \delta(\omega_0 - 1)\delta(\omega_1 - i)b_0(\omega_2, \dots) + \delta(\omega_0 + 1)\delta(\omega_1 - i)b^*(-\omega_2, \dots). \quad (3.13)$$

The non-secular solution for  $g_1$  that satisfies

$$(\omega_0^2 - 1)g_1(\omega_0, \omega_1, \dots) = 0, \quad (3.14)$$

is provided by

$$g_1(\omega_0, \omega_1, \dots) = \delta(\omega_0 - 1)a_1(\omega_1, \dots) + \delta(\omega_0 + 1)a_1^*(-\omega_1, \dots), \quad (3.15)$$

which is equivalent to the solution (2.11) in the physical space representation.

Introducing (3.13) and (3.15) into (3.5c), we obtain

$$\begin{aligned} (\omega_0^2 - 1)g_2(\omega_0, \omega_1, \dots) = & 2\omega_0(\omega_1 - i) [\delta(\omega_0 - 1)a_1(\omega_1, \dots) + \delta(\omega_0 + 1)a_1^*(-\omega_1, \dots)] \\ & + (\omega_1^2 + 2\omega_0\omega_2 - 2i\omega_1) [\delta(\omega_0 - 1)\delta(\omega_1 - i)b_0(\omega_2, \dots) + \delta(\omega_0 + 1)\delta(\omega_1 - i) \\ & \times b_1^*(-\omega_2, \dots)]. \end{aligned} \quad (3.16)$$

Now, singularities arise at  $\omega_0 = \pm 1$  when we solve (3.16) for  $g_2$ , but they can be eliminated if we can set

$$\left. \begin{aligned} 2(\omega_1 - i)a_1(\omega_1, \dots) + (1 + 2\omega_2)\delta(\omega_1 - i)b(\omega_2, \dots) &= 0, \\ -2(\omega_1 - i)a_1^*(-\omega_1, \dots) + (1 - 2\omega_2)\delta(\omega_1 - i)b_0^*(-\omega_2, \dots) &= 0, \end{aligned} \right\} \quad (3.17)$$



This relation corresponds to (2.12) in the physical space representation.

When we solve (3.17) for  $a_1$ , a singularity arises at  $\omega_1 = i$ , because of the appearance of the factor  $\delta(\omega_1 - i)/(\omega_1 - i)$ . To remove such singularities we can choose

$$(1 + 2\omega_2)b_0(\omega_2, \dots) = 0, \quad (1 - 2\omega_2)b_0^*(-\omega_2, \dots) = 0, \quad (3.18)$$

that is,

$$b_0(\omega_2, \omega_3, \dots) = \delta(\omega_2 + \frac{1}{2})c_0(\omega_3, \dots), \quad b^*(-\omega_2, -\omega_3, \dots) = \delta(\omega_2 - \frac{1}{2})c^*(-\omega_3, \dots). \quad (3.19)$$

Then  $a_1(\omega_1, \dots)$  is given by

$$\begin{aligned} a_1(\omega_1, \dots) &= \delta(\omega_1 - i)b_1(\omega_2, \dots), \\ a_1^*(-\omega_1, \dots) &= \delta(\omega_1 - i)b_1^*(-\omega_2, \dots). \end{aligned} \quad (3.20)$$

The inverse Fourier transformations of (3.19) and (3.20) reproduce the physical space representations (2.16) and (2.17).

Using (3.19) and (3.11) in (3.7), we find

$$g_0(\omega_0, \omega_1, \dots) = \delta(\omega_0 - 1)\delta(\omega_1 - i)\delta(\omega_2 + \frac{1}{2})c_0(\omega_3, \dots) + \delta(\omega_0 + 1)\delta(\omega_1 - i)\delta(\omega_2 - \frac{1}{2}) \times c_0^*(-\omega_3, \dots). \quad (3.21)$$

The inverse Fourier transformation of (3.21) gives

$$C_0(t_3, \dots) e^{it_0 - it_2/2} e^{-t_1} + C_0^*(t_3, \dots) e^{-it_0 + it_2/2} e^{-t_1}, \quad (3.22)$$

where  $C_0(t_3, \dots)$  represents the inverse Fourier transformation of  $c_0(\omega_3, \dots)$ .

The next order problem is given by

$$\begin{aligned} (-\omega_0^2 + 1)g_3 &= 2\delta(\omega_0 - 1)[(\omega_1 - i)a_2 + \delta(\omega_1 - i)(\omega_2 + \frac{1}{2})b_1 + \delta(\omega_1 - i)\delta(\omega_2 + \frac{1}{2})\omega_3 c_0] \\ &\quad - 2\delta(\omega_0 + 1)[(\omega_1 - i)a_2^* + \delta(\omega_1 - i)(\omega_2 - \frac{1}{2})b_1^* + \delta(\omega_1 - i)\delta(\omega_2 + \frac{1}{2})\omega_3 c_0^*]. \end{aligned} \quad (3.23)$$

The non-secular condition for  $g_3$  is

$$(\omega_1 - i)a_2(\omega_1, \dots) + \delta(\omega_1 - i)(\omega_2 + \frac{1}{2})b_1(\omega_2, \dots) + \delta(\omega_1 - i)\delta(\omega_2 + \frac{1}{2})\omega_3 c_0(\omega_3, \dots) = 0, \quad (3.24)$$

together with a similar relation involving asterisks. The relation (3.24) corresponds to (2.18) in the physical space representation. When we solve (3.24) for  $a_2$ , there arises the singularity  $\delta(\omega_1 - i)/(\omega_1 - i)$ . It can be removed if we set

$$(\omega_2 + \frac{1}{2})b_1(\omega_2, \dots) + \delta(\omega_2 + \frac{1}{2})\omega_3 c_0(\omega_3, \dots) = 0. \quad (3.25)$$

Similarly, a singularity related to  $\delta(\omega_2 + \frac{1}{2})/(\omega_2 + \frac{1}{2})$  arises when seeking  $b_1$ . It also can be avoided if we set

$$\omega_3 c_0(\omega_3, \dots) = 0, \quad (3.26a)$$

or, if we set

$$c_0(\omega_3, \omega_4, \dots) = \delta(\omega_3)d_0(\omega_4, \dots). \quad (3.26b)$$

We can proceed with similar calculations up to any order of approximation in a straightforward manner. For example, up to the  $O(\epsilon^4)$ -approximation, we obtain

$$\begin{aligned}
g &= g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + \varepsilon^3 g_3 + \varepsilon^4 g_4 \\
&= \delta(\omega_0 - 1) \left[ \delta(\omega_1 - i) \delta(\omega_2 + \tfrac{1}{2}) \delta(\omega_3) \delta(\omega_4 + \tfrac{1}{8}) e_0 + \varepsilon \delta(\omega_1 - i) \delta(\omega_2 + \tfrac{1}{2}) \delta(\omega_3) d_1 \right. \\
&\quad \left. + \varepsilon^2 \delta(\omega_1 - i) \delta(\omega_2 + \tfrac{1}{2}) c_2 + \varepsilon^3 \delta(\omega_1 - i) b_3 + \varepsilon^4 a_4 \right] + \delta(\omega_0 + 1) \\
&\quad \times \left[ \delta(\omega_1 - i) \delta(\omega_2 - \tfrac{1}{2}) \delta(\omega_3) \delta(\omega_4 - \tfrac{1}{8}) e_0^* + \varepsilon \delta(\omega_1 - i) \delta(\omega_2 - \tfrac{1}{2}) \delta(\omega_3) d_1^* + \varepsilon^2 \delta(\omega_1 - i) \right. \\
&\quad \left. \times \delta(\omega_2 - \tfrac{1}{2}) c_2^* + \varepsilon^3 \delta(\omega_1 - i) b_3^* + \varepsilon^4 a_4^* \right]. \quad (3.27)
\end{aligned}$$

The use of the boundary conditions for this solution, together with the assumption that  $e_0$ ,  $d_1$ ,  $c_2$ ,  $b_3$ , and  $a_4$  are constants with respect to the slow variables equal to or slower than  $\omega_5$ ,  $\omega_4$ ,  $\omega_3$ ,  $\omega_2$ , and  $\omega_1$ , respectively, leads to a solution equivalent to (2.21) in the physical space representation. Thus, once the delta function in a complex Fourier space has been introduced, we can carry out the perturbation analysis systematically in the Fourier space.

#### 4. CONCLUDING REMARKS

Sandri [4] applied the multiple scale Fourier transformation to a problem involving exponential changes. However, there was an error in his paper, since he omitted the imaginary unit in his calculations on page 91. Because of this error, he was unaware of the point discussed in this paper that the delta function should be extended to a complex space, corresponding to the Fourier transformation of an exponential function.

If the Fourier transformation is extended to a complex space on a mathematically sound basis, we can apply the multiple scale Fourier transformation to perturbation problems in general physical systems, merely keeping in mind that the multiple scales in the Fourier space are complex quantities. In practical calculations, the use of the delta functions in the complex space may simplify the manipulations associated with the perturbation analysis.

Although the example treated in this paper is simple, the essential point involved is well illustrated. The extension presented in this paper will benefit the Fourier transformation method when it is applied to solve general wave problems which involve dissipative effects.

*Acknowledgements*—This research was supported in part by the Air Force Office of Scientific Research, the Office of Naval Research, the Army Research Office and National Science Foundation while one of the authors (AJ) was a Visiting Scholar in the Applied Mathematics Group, Stanford University.

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#### APPENDIX A. FOURIER TRANSFORMATION OF EXPONENTIAL FUNCTIONS

Let us denote the Fourier transformation of  $f$  by

$$\mathcal{F}(f, \omega) = g(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt, \quad (\text{A.1})$$

and define its inverse by

$$\mathcal{F}^{-1}(g, t) = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) e^{-i\omega t} d\omega. \quad (\text{A.2})$$

When  $f$  and  $g$  belong to the class of  $L_1$  or  $L_2$ -functions, their Fourier transforms exist in the ordinary sense. If this is not the case, we must then introduce the notion of generalized functions together with an extended definition for the Fourier transformation. Such generalized Fourier transformations can be introduced in terms of those for distributions (Bremermann [2], Challifour [3]) or those for hyper-functions (Imai [5]).

The distribution generated by  $f(t)$  is defined by

$$\langle f, \phi \rangle = \int_{-\tau}^{\infty} f(t) \phi(t) dt, \quad (\text{A.3})$$

where  $\phi$  is a test function. Let  $T$  be a distribution, then its Fourier transformation is introduced as follows

$$\langle \mathcal{F}(T), \phi \rangle = \langle T, \mathcal{F}(\phi, t) \rangle, \quad (\text{A.4})$$

when the inversion theorem becomes

$$\langle \mathcal{F}(\mathcal{F}^{-1}(T)), \phi \rangle = \langle \mathcal{F}^{-1}(\mathcal{F}(T)), \phi \rangle = \langle T, \phi \rangle. \quad (\text{A.5})$$

As an example, let us consider that 1 denotes the distribution generated by the unit function 1. Then it follows that

$$\langle \mathcal{F}(\delta), \phi \rangle = \langle \delta\omega, e^{i\omega t} \rangle, \phi(t) = 1. \quad (\text{A.6})$$

Thus we have  $\mathcal{F}(\delta) = 1$ . By means of (A.5) we have  $\mathcal{F}^{-1}(1) = \delta$ . Furthermore, it is easily shown that  $\mathcal{F}^{-1}(\delta) = \frac{1}{2\pi}$  and  $\mathcal{F}(1) = 2\pi\delta$ .

Let us now consider the Fourier transformation of  $e^{ia\omega}$ . Using the formulae (A.3) to (A.5), we have

$$\begin{aligned} \langle \mathcal{F}(e^{ia\omega}), \phi(\omega) \rangle &= \langle e^{ia\omega}, \mathcal{F}(\phi, t) \rangle \\ &= \left\langle 1, \int_{-\infty}^{\infty} \phi(\omega) e^{it(\omega+a)} d\omega \right\rangle \\ &= \left\langle 1, \int_{-\infty}^{\infty} \phi(\omega - a) e^{i\omega t} d\omega \right\rangle \\ &= \langle 2\pi\delta, \phi(\omega - a) \rangle = 2\pi\phi(-a) \\ &= \langle 2\pi\delta(\omega + a), \phi(\omega) \rangle. \end{aligned} \quad (\text{A.7})$$

Hence we have

$$\mathcal{F}(e^{ia\omega}, \omega) = 2\pi\delta(\omega + a). \quad (\text{A.8})$$

The formula (A.8) has been obtained for real  $a$  in the above proof. We may also give a similar proof for a complex  $\omega$ , in order to include the Fourier transformation for an exponential function, say  $e^{bt}$ . It is possible to choose the test function  $\phi$  such that it vanishes outside a compact set, and thus the integral of the product  $\phi e^{i\omega t}$  with  $e^{bt}$  becomes bounded along the appropriate integration path in the complex space. In such a way, the Fourier transformation of  $e^{bt}$  may be given as

$$\mathcal{F}(e^{bt}, \omega) = 2\pi\delta(\omega + bi). \quad (\text{A.9})$$

Consequently, we arrive at the formulae (3.12) given in the text.

The Fourier transformation of generalized functions can also be introduced in terms of hyper-function theory (Imai [5]). We do not go into details here, but notice only that the formulae (3.12) are also obtainable in terms of the Fourier transformation of hyper-functions.

## APPENDIX B. APPLICATION OF THE DELTA FUNCTION IN A COMPLEX SPACE

Let us solve the problem (2.1) subject to (2.2) in a Fourier space, assuming that  $\varepsilon$  is of order unity. We consider the Fourier transformation

$$f(t) = \int_{\gamma} g(\omega) e^{i\omega t} d\omega, \quad (\text{B.1})$$

where  $\omega$  is extended to a complex space, and  $\gamma$  is a suitable contour.

Introducing (B.1) into (2.1) and (2.2), we obtain

$$(-\omega^2 + 2i\varepsilon\omega + 1)g(\omega) = 0, \quad (\text{B.2})$$

$$\int_{\gamma} g(\omega) d\omega = a, \quad \int_{\gamma} i\omega g(\omega) d\omega = 0. \quad (\text{B.3})$$

When we admit delta functions in the complex  $\omega$ -space, the general solution of (B.2) can be expressed as follows:

$$g(\omega) = \delta(\omega - \sqrt{1 - \varepsilon^2} - i\varepsilon)C + \delta(\omega + \sqrt{1 - \varepsilon^2} - i\varepsilon)D. \quad (\text{B.4})$$

The inverse Fourier transformation then gives

$$\begin{aligned} f(t) &= \int_{\gamma} \delta(\omega - \sqrt{1 - \varepsilon^2} - i\varepsilon)C e^{i\omega t} d\omega \\ &\quad + \int_{\gamma} \delta(\omega + \sqrt{1 - \varepsilon^2} - i\varepsilon)D e^{i\omega t} d\omega \\ &= e^{-\varepsilon t} [C \exp i\sqrt{1 - \varepsilon^2} t + D \exp -i\sqrt{1 - \varepsilon^2} t]. \end{aligned} \quad (\text{B.5})$$

The boundary conditions give the relations

$$C + D = a, \quad i\sqrt{1 - \varepsilon^2} (C - D) - \varepsilon(C + D) = 0, \quad (\text{B.6})$$

which, in turn, give rise to

$$\begin{aligned} C &= \frac{1}{2} \left( 1 - \frac{i\varepsilon}{\sqrt{1 - \varepsilon^2}} \right) a, \\ D &= \frac{1}{2} \left( 1 + \frac{i\varepsilon}{\sqrt{1 - \varepsilon^2}} \right) a. \end{aligned}$$

Thus we obtain the exact solution (2.3) quite easily in terms of a delta function in a complex Fourier space.



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